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ON GROUP RINGS OF NILPOTENT GROUPS

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ABSTRACT
It is shown that if G is a non-abelian torsion free nilpotent group and F is a field, then the classical skew field of fractions $F(G)$ of the group ring $F[G]$ contains a noncommutative free subalgebra.

As is well known, if G is a torsion free nilpotent group and F is a field, then the group algebra $F[G]$ is an Ore ring and has a uniquely defined skew field of fractions [6]. It is clear that this algebra does not contain free subalgebras of rank bigger than one. Nevertheless, it is true that its skew field of fractions $F(G)$ contains free subalgebras of larger rank, if, of course, G is nonabelian.

We are going to use the following two facts (see, e.g., Theorems 1 and 2 in [4]).

LEMMA 1. *Every nilpotent nonabelian group G contains a subgroup H with generators a and b for which $(a,b)=aba^{-1}b^{-1}=c\neq 1$ and $(a,c)=(b,c)=1$.*

LEMMA 2. *If H is a torsion-free group generated by a and b such that $(a,b)=c$ where c commutes with a and b then $H/(c)$ is a free abelian group of rank two. (Here (c) is the normal subgroup spanned by c.)*

THEOREM. *If H is the group of Lemma 2 then the skew field $F(H)$ of fractions of $F[H]$ contains a free subalgebra of rank two.*

PROOF. We are going to show that the elements $(1-a)^{-1}$ and $(1-a)^{-1}(1-b)^{-1}$ generate a free subalgebra. As the first step let us take the algebra L which consists of $\sum_{i=k}^{\infty} b^i r_i(a)$ where $r_i \in F(c)(a)$ and k can be negative. The operations in L are standard addition and convolution which respects relation $ab=cba$, namely

$$\sum b^i r_i(a)\cdot \sum b^j s_j(a)=\sum b^{i+j} r_i(c^j a) s_j(a).$$

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It is clear that multiplication is associative and that L is a skew field. It is also clear that the skew subfield of L generated by a, b and c is isomorphic to the skew field $F(H)$.[†]

Now, using reasoning similar to that in [5], we are going to show that the monomials

$$m_I=(1-a)^{-i_0}(1-b)^{-1}(1-a)^{-i_1}\cdots(1-a)^{-i_{k+1}}$$

are linearly independent over F.

In L we have $(1-b)^{-1}=\Sigma_{i=0}^{\infty}b^i$, so m_I can be written as

$$m_I=\sum_J b^{j_0+\cdots+j_k}(1-c^{j_0+\cdots+j_k}a)^{-i_0}(1-c^{j_1+\cdots+j_k}a)^{-i_1}\cdots(1-a)^{-i_{k+1}}.$$

If we denote

$$\sum_{n\geq n_1\geq\cdots\geq n_k\geq 0}(1-c^{n_1}a)^{-i_1}\cdots(1-c^{n_k}a)^{-i_k}$$

by $f_I(n)$ (here $i_s\neq 0$), then

$$m_I=\sum_n b^n(1-c^na)^{-i_0}(1-a)^{-i_{k+1}}f_I(n).$$

Let us assume that m_I are linearly dependent: $\Sigma_I g_I m_I=0$ for some $g_I\in F$. Then

$$\sum_n b^n\Big(\sum_I g_I(1-c^na)^{-i_0}(1-a)^{-i_{k+1}}f_I(n)\Big)=0$$

which means that

$$\sum_I g_I(1-c^na)^{-i_0}(1-a)^{-i_{k+1}}f_I(n)=0$$

for each natural number n.

Let us consider now all possible dependences $\Sigma\,\lambda_I^{(n)}f_I=0$ for $f_I(n)$, where $\lambda_I^{(n)}$ is obtained from a rational function $\lambda_I\in F(a,c,x)$ by substituting c^n instead of x. Among these find a minimal dependence, that is, a dependence with the following properties:

(a) the number of coordinates in the index vectors I for f_I involved is minimal,

(b) the number of functions with index vectors of this maximal length is minimal.

(We are choosing a dependence relation which satisfies (a) and then (b).)

[†] This follows, e.g., from Proposition 1.2.3 in [1].

$$\sum c_I f_I = \sum d_J f_J. \tag{1}$$

Here all "long" vectors are collected on the left side.

Let us consider now the relation

$$\sum \Delta(c_I f_I) = \sum \Delta(d_J f_J) \tag{2}$$

where $\Delta g(n) = g(n+1) - g(n)$, so for example $\Delta c^n = c^{n+1} - c^n = (c-1)c^n$.

It is easy to see that

$$\Delta f_I(n) = \sum_{m=1}^{k} f_{I(m)}(n) \cdot (1 - c^{n+1}a)^{-i_1 - i_2 - \cdots - i_m}$$

where $I(m) = \{i_{m+1}, i_{m+2}, \ldots, i_k\}$.

Without loss of generality we can assume that one of the coefficients on the left side of (1) is equal to one. Now we can use the identity

$$\Delta(g(n)f(n)) = (\Delta g(n))f(n) + g(n+1)\Delta f(n)$$

and rewrite (2) as

$$\sum (\Delta c_I) f_I + \sum c_I(n+1)\Delta f_I = \sum (\Delta d_J) f_J + \sum d_J(n+1)\Delta f_J.$$

So (2) has fewer terms than (1) with long vectors. This means that all coefficients of f's in (2) should be zeros. Therefore $\Delta c_I = 0$ for all c_I, and hence $c_I \in F(a, c)$, and

$$\sum c_I (1 - c^{n+1}a)^{-i_1} - \Delta d_J = 0$$

where the summation runs over all I with the same $I(1)$ and $J = I(1)$. Thus

$$d_J(a, c \cdot c^n) - d_J(a, c^n) = \sum c_I(a, c)(1 - cc^n a)^{-i_1}. \tag{3}$$

In both sides of (3) we have rational expressions in a, c and $c^n = x$. If we regard a and c as parameters then equality (3) holds for infinitely many values of x. Now for rational functions this means that they are equal for every value of x (since a polynomial cannot have too many zeros). So (3) is valid for every value of x from $F(a, c)$. The right side of (3) has a singularity only at the point $x = c^{-1}a^{-1}$. This means that $d_J(a, x) = d(x)$ has a singularity at $x = c^{-1}a^{-1}$ or at $x = a^{-1}$. If $d(x)$ has a singularity at $x = c^{-1}a^{-1}$ then $d(x)$ has singularities at all points $c^{-2}a^{-1}, c^{-3}a^{-1}, \ldots$; if $d(x)$ has a singularity at $x = a^{-1}$ then $d(x)$ has singularities at all points $ca^{-1}, c^2a^{-1}, \ldots$ because otherwise the right side of (3) would have a singularity at one of these points. But a rational function cannot have infinitely many singularities, so (3) is impossible and so is (1). The theorem is proved.

COROLLARY. *If G is a torsion free nonabelian nilpotent group and F is a field then the skew field $F(G)$ contains a free subalgebra of rank two and so of every countable rank.*

PROOF. By Lemma 1, the group G contains a subgroup H which satisfies the conditions of the Theorem. Obviously the skew field $F(G)$ contains the skew field $F(H)$ (see the previous footnote) and so by the Theorem, also contains a free subalgebra of rank two. But, as is well known, such an algebra contains a free subalgebra of every countable rank.

REMARK 1. Clearly the result is true for locally nilpotent groups because any such group contains a subgroup H as above.

REMARK 2. By a result of M. Gromov [3], the group ring of a finitely generated group G has a finite Gelfand–Kirillov dimension [2] if and only if G is finite-by-nilpotent. From this result and from the Corollary it follows that if G is a finitely generated group for which $F(G)$ exists and the Gelfand–Kirillov dimensions of the subalgebras of $F(G)$ are uniformly bounded then G is finite-by-abelian.

REMARK 3. Suppose a skew field D over a field F contains two elements a and b such that $c = (a, b)$ commutes with both a and b, and c has infinite multiplicative order. One can see that the proof of the Theorem above does not depend on the assumption that c is transcendental over F but only on the assumption that the sequence $\{c^n\}$ contains infinitely many different terms. Thus the skew subfield generated by a and b over F contains a free algebra.

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